CHAPTER 7

On Commodity Prices and Factor Rewards: A Close Look at Sign Patterns

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Abstract
The effect of changes in commodity prices on factor rewards is studied in the multi-commodity, multi-factor case. It is shown that the inverse of the distributive share matrix must satisfy the following restriction: it cannot be *anti-symmetric in its sign pattern*. This means that one cannot partition the commodities into two groups (I and II) and factors into two groups (A and B), such that all factors in group A benefit (nominally) from all commodity price increases in group I, and simultaneously all factors in group B suffer from all commodity price increases in group II. It turns out that this is also the *only* sign-pattern restriction imposed by the general nature of the relationship of commodity prices and factor rewards.

Keywords: Stolper-Samuelson result, factor rewards, distributive share matrix, sign-pattern restriction

JEL classifications: C65, C67, D33, D51, F11

1. Introduction

The relation between commodity prices and factor rewards is an enduring theme in the contributions of Ron Jones to the pure theory of international trade. From his famous 1965 paper, "On the Structure of Simple General Equilibrium Models" (Jones, 1965) to his 2006 expository piece, "'Protection and Real Wages': The History of an Idea," (Jones, 2006) we see in his research on this topic a sustained interest that is truly remarkable. Inspired by the original Stolper–Samuelson (1941) paper, this topic came to represent for him the essence of the applicability of international trade theory to policy issues, as well as a leading test case of the importance of general equilibrium analysis.

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If one follows the development of his ideas on this topic over this 40-year period, one can discern at least two distinct themes.[1] The first is concerned primarily with the Stolper–Samuelson *result*, and its generalizations to the case of many commodities and many factors. This line of his research has identified *particular* production structures, in the many commodity, many factor case, for which the Stolper–Samuelson result continues to hold; they include the "produced mobile factor structure" discussed in Jones (1975) and Jones and Marjit (1985), and the "neighborhood production structure" analyzed in Jones and Kierzkowski (1986).[2] It has also identified *sufficient conditions* on the share matrix for *general* production structures which ensure the Stolper–Samuelson result in either its strong form (see Jones *et al*., 1993) or its weak form (see Mitra and Jones, 1999).[3]

The second theme is concerned with the Stolper–Samuelson *idea* that the impact of changes in commodity prices on factor rewards can be studied by systematic use of general equilibrium analysis. In this theme, the emphasis is less on the original Stolper–Samuelson result, and more on identifying useful predictions that can be made about changes in the distribution of income, following changes in commodity prices. Pioneered by Ethier (1974), this line of research has been pursued by, among others, Kemp and Wan (1976), Jones and Scheinkman (1977), Chang (1979) and Jones (1985).[4] While appropriate conditions on factor intensities figure prominently in the first theme (following, notably, the contributions of Chipman (1969) and Kemp and Wegge (1969)), they play no role in the second.

The results I will be discussing in this chapter belong to the second theme. We consider the setting which involves the active production in competitive markets of n commodities, each produced non-jointly by the use of n distinct factors of production (known as the "even case") in processes that are linearly independent of each other at prevailing factor prices. The distributive share of factor i in industry j denoted by α_{ij} is assumed to be strictly positive for all i, j.[5]

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[1] To be sure, this statement is a gross oversimplification. But, since it helps me to a certain extent in fitting my own contribution in his scheme of ideas, I will maintain it.

[2] The analysis of the shape of "share ribs" in Jones and Mitra (1995) was used to unify these results in a common (but still special) production structure.

[3] These conditions, though strong, can be directly verified using only the information about the share matrix, and are to be distinguished in this respect from more general conditions, which have been proposed in the literature, but which are not verifiable in the same way; see, especially, the well-known papers by Uekawa (1971) and Inada (1971).

[4] For useful surveys of the relevant literature, we refer the reader to Ethier (1984) and Jones and Neary (1984).

[5] The equality of the number of factors and the number of commodities, as well as the positivity of all α_{ij}, are strong assumptions of this framework. More general frameworks, in which the number of commodities need not be equal to the number of factors, and some of the α_{ij} can be zeroes, have been studied by, among others, Jones and Scheinkman (1977) and Chang (1979).

The inverse of the column-stochastic matrix (α_{ij}) denoted by (β_{ij}), where $\beta_{ij} = [\partial w_j(p)/\partial p_i][p_i/w_j]$, is a column-stochastic matrix which provides the full information of the effects of commodity prices (p_i) on factor rewards (w_j). It can be seen easily that the matrix (β_{ij}) must satisfy the restriction (referred to as (R) in Section 3) that every row and every column of it must have both positive and negative elements. The implications that have been drawn from this information can be summarized as follows:

(i) For each factor, there is some commodity, such that (*ceteris paribus*) an increase in price of that commodity leads to a decrease in the factor's reward.
(ii) For every commodity, there is some factor, such that (*ceteris paribus*) an increase in price of the commodity leads to a decrease in that factor's reward.
(iii) For every commodity, there is some factor, such that (*ceteris paribus*) an increase in price of the commodity leads to an increase in that factor's *real* reward.
(iv) For each factor, there is some commodity, such that (*ceteris paribus*) an increase in price of that commodity leads to a *nominal* increase in the factor's reward.
(v) For each factor, there is some *set* of commodities such that an equiproportionate increase in the prices of all commodities in that set, other prices being constant, will increase the *real reward* of the factor.

We now ask whether these implications characterize the qualitative information that can be derived regarding the relationship of commodity prices to factor rewards in this framework. Apparently not, because the matrix

$$T = \begin{bmatrix} 1.5 & -1 & -1 & -1 \\ -1 & 1.5 & -1 & -1 \\ 0.25 & 0.25 & 4 & -1 \\ 0.25 & 0.25 & -1 & 4 \end{bmatrix}$$

is a column-stochastic matrix, which satisfies restriction (R) and all the properties listed in (i)–(v), but T *cannot* be generated as the inverse of any distributive share matrix (in our framework). In other words, the dependence of factor rewards on commodity prices exhibiting the behavior displayed in T cannot be rationalized in terms of our framework.

It is possible to reach this conclusion without performing extensive calculations, but by simply observing the *sign pattern* of the matrix T, and applying Theorem 1, stated in Section 3 of this chapter. But, the reader is asked to refrain from skipping ahead to Section 3, and encouraged to try to figure out this conclusion based on the information provided above.[6]

Theorem 1 in this chapter develops a condition (referred to as Condition C in Section 3) on the sign pattern of T which *must be violated* if T is to be the inverse of a distributive share matrix. Further, given the restriction (R), the violation of Condition C captures all the qualitative restrictions imposed by our framework on the relationship between commodity prices and factor rewards. This is reported in Theorem 2.

The principal use of Theorem 1 is to rule out some kinds of behavior of factor rewards following commodity price changes, as illustrated in the above example of T, which are not already ruled out by the restriction (R). The principal use of Theorem 2 is that it gives us an easy way to rationalize a variety of behavior of factor reward changes following commodity price changes.

2. *Preliminaries*

2.1. Notation

For vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$ for $i = 1, \ldots, n$; $x > y$ means $x \geq y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for $i = 1, \ldots, n$. The set $\{x \in \mathbb{R}^n : x \geq 0\}$ is denoted by $\mathbb{R}^n_+$; the set $\{x \in \mathbb{R}^n : x \gg 0\}$ is denoted by Ω.

The n unit vectors in $\mathbb{R}^n$ are denoted by $e^1, \ldots, e^n$; the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$ is denoted by u.

Let $A = (a_{ij})$ be an $n \times n$ real matrix. Then A is called a *positive matrix* if $a_{ij} > 0$ for all $i, j = 1, \ldots, n$. It is called a *positive diagonal matrix* if it is a diagonal matrix, and $a_{ii} > 0$ for all $i = 1, \ldots, n$.

2.2. The framework

Consider an n-commodity, n-factor model of production (with $n \geq 2$), in which y_j denotes the output of the jth commodity and x_{ij} the amount of the ith factor used in the production of the jth commodity (where $i, j = 1, 2, \ldots, n$).

For each commodity j, the output level y_j is determined by input levels $(x_{1j}, \ldots, x_{nj})$ of the n factors, according to a production function f_j from $\mathbb{R}^n_+$ to $\mathbb{R}_+$:

$$y_j = f_j(x_{1j}, \ldots, x_{nj}) \quad \text{for } j = 1, \ldots, n. \tag{1}$$

[6] I must confess that, to me, the conclusion is not immediately obvious, and not obvious even after a great deal of thought.

For each $j = 1, \ldots, n$, assume:[7]

(F.1) f_j is non-decreasing, continuous and homogeneous of degree one on $\mathbb{R}^n_+$.
(F.2) $f_j(x) > 0$ if and only if $x \gg 0$.
(F.3) f_j is strictly quasi-concave on Ω.

For each commodity j, given an output level $y > 0$, and a vector of factor prices $w = (w_1, \ldots, w_n) \gg 0$, consider the cost-minimizing problem:

$$\left.\begin{array}{ll} \text{Minimize} & wx \\ \text{subject to} & f_j(x) \geq y \\ \text{and} & x \in \mathbb{R}^n_+ \end{array}\right\} \text{(CM)}.$$

Problem (CM) has a unique solution, $x_j(w, y) = (x_{1j}(w, y), \ldots, x_{nj}(w, y))$ in Ω. These are the *conditional input demands* in the production of commodity j. The *cost* function $c_j(w, y)$ is then $wx_j(w, y)$. Since the production function f_j is homogeneous of degree one, we have for each j,

$$c_j(w, y) = yc_j(w, 1); \quad x_j(w, y) = yx_j(w, 1).$$

Denote the *unit cost* functions $c_j(w,1)$ by $g_j(w)$, and the *unit conditional demand* functions $x_j(w,1)$ by $a_j(w) = (a_{1j}(w), \ldots, a_{nj}(w))$. Thus, for each j,

$$g_j(w) = \sum_{i=1}^{n} a_{ij}(w) w_i. \tag{2}$$

It is known that for $i, j = 1, \ldots, n$, the functions $a_{ij}(w)$ are continuous on Ω. Also, for each $j = 1, \ldots, n$, the function $g_j(w)$ has continuous partial derivatives on Ω, and we have ("Shephard's Lemma"):[8]

$$\frac{\partial g_j(w)}{\partial w_i} = a_{ij}(w) \quad \text{for } i = 1, \ldots, n. \tag{3}$$

If the n commodities are traded in a competitive market under a price system $p \in \Omega$, the price of the jth commodity must equal its unit cost $g_j(w)$ if it is positively produced in equilibrium. Proceeding under the assumption that given the price vector $p^0 \in \Omega$, the economy is in an incomplete specialization equilibrium at factor prices $w^0 \in \Omega$, we then have

$$\left.\begin{array}{l} \text{(i) } a_{ij}(w^0) > 0 \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n \\ \text{(ii) } g_j(w^0) = p_j^0 \quad \text{for } j = 1, \ldots, n \end{array}\right\}. \tag{4}$$

Assume that the Jacobian of $g(w) = (g_1(w), \ldots, g_n(w))$ is non-zero at $w = w^0$. Then, by the inverse function theorem, there is a neighborhood

[7] The assumptions, in fact, imply that each f_j is concave on $\mathbb{R}^n_+$.
[8] See, especially, Nikaido (1968, p. 357).

U of w^0 and a neighborhood V of p^0, and a unique mapping $h : U \to V$ satisfying

$$h(g(w)) = w \text{ for all } w \in U. \tag{5}$$

Furthermore, h has continuous partial derivatives on V, and

$$[Dh(p^0)] = [Dg(w^0)]^{-1}, \tag{6}$$

where $[Dg(w^0)]$ is the Jacobian matrix of g at w^0 and $[Dh(p^0)]$ the Jacobian matrix of h at p^0 (See, especially, Apostol, 1957, p. 144).

If we denote the positive matrix $(a_{ij}(w^0))$ by $A \equiv A(w^0)$, then by (3) and (6) respectively, we have

$$[Dg(w^0)]' = A \tag{7}$$

and

$$[Dh(p^0)]' = A^{-1} = (b_{ij}). \tag{8}$$

Recalling (5), we can rewrite (8) as

$$\left[\frac{\partial w_j(p^0)}{\partial p_i}\right] = (b_{ij}) = A^{-1}. \tag{9}$$

Define the matrix of *distributive shares*, $S = (\alpha_{ij})$ by

$$\alpha_{ij} = \frac{w_i^0 a_{ij}(w^0)}{p_j^0} \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n. \tag{10}$$

Then, by using (2) and (4), we have

$$\left.\begin{array}{ll} \text{(i)} & \alpha_{ij} > 0 \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n \\ \text{(ii)} & \sum_{i=1}^{n} \alpha_{ij} = 1 \quad \text{for each } j = 1, \ldots, n \end{array}\right\}. \tag{11}$$

That is, each column sum of S is equal to 1.

Define P to be the diagonal matrix with $(p_1^0, \ldots, p_n^0)$ on its diagonal; define W to be the diagonal matrix with $(w_1^0, \ldots, w_n^0)$ on its diagonal. Then, (10) implies that

$$S = WAP^{-1}. \tag{12}$$

So, S has an inverse, and denoting the inverse of S by T, we have

$$T = PA^{-1}W^{-1}. \tag{13}$$

Denoting by (β_{ij}) the matrix T, we observe from (9) and (13),

$$\beta_{ij} = \left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n. \tag{14}$$

We are principally interested in making predictions about β_{ij}.

Denoting the column vector (1, 1, …, 1) in $\mathbb{R}^n$ by u, we observe that by (11), we have

$$u'S = u'. \tag{15}$$

So, post-multiplying both sides of (15) by T, we get

$$u' = u'T. \tag{16}$$

That is, each column sum of T is equal to 1, and consequently,

$$\sum_{i=1}^{n} \beta_{ij} = \sum_{i=1}^{n} \left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] = 1 \quad \text{for each } j = 1, \ldots, n. \tag{17}$$

In what follows, we suppress the factor prices w^0 and commodity prices p^0 at which our analysis is carried out. Thus, our analysis is explicitly *local* in a neighborhood of the equilibrium given by (p^0, w^0).

3. Effects of commodity price changes on factor rewards

3.1. Review of the literature

We review what is known about the effect of commodity prices on factor returns, that is about β_{ij}. Since

$$ST = TS = I \tag{18}$$

and S is a positive matrix, we have the following restriction:

(R) Every column and every row of T must have positive as well as negative entries.[9]

One can elaborate on the implications of the restriction (R) as follows:

(i) For each j, there is some i such that

$$\beta_{ij} = \left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] < 0;$$

that is, for each factor j, there is some commodity i such that an increase in price of commodity i alone leads to a decrease in factor j's reward.[10]

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[9] If the ith column of T has only non-positive elements, then $S_i T^i \leq 0$, since S is a non-negative matrix. Here S_i is the ith row of S and T^i is the ith column of T. But, by (18), $S_i T^i = 1$, a contradiction.

Suppose the ith column of T has only non-negative elements. Using (18), we have $S_j T^i = 0$ for $j \neq i$, and so the ith column of T has only zero elements, since S is a positive matrix. Thus, we must have $S_i T^i = 0$. But, by (18), $S_i T^i = 1$, a contradiction.

Analogous arguments with the rows of T yield the remaining results claimed in (R).

[10] To use the terminology of Ron Jones, each factor has a commodity enemy.

(ii) For each i, there is some j such that

$$\beta_{ij} = \left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] < 0;$$

that is, for every commodity i, there is some factor j such that an increase in price of commodity i alone leads to a decrease in factor j's reward.[11]

(iii) For each i, there is some j such that

$$\beta_{ij} = \left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] > 1;$$

that is, for every commodity i, there is some factor j such that an increase in price of commodity i alone leads to an increase in factor j's *real* reward.[12]

(iv) For each j, there is some i such that

$$\beta_{ij} = \left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] > 0;$$

that is, for each factor j, there is some commodity i such that an increase in price of commodity i alone leads to a *nominal* increase in factor j's reward. It is important to note that this conclusion is not symmetric to (iii). The *real* reward of factor j can go down, since the price of some commodity has gone up.[13]

If one combines implication (i) with (17), one may, however, note a further implication (this is due to Jones (1985)).

(v) For each j, there is some set $F \subset \{1, 2, \ldots, n\}$, such that

$$\sum_{i\in F}\beta_{ij} = \sum_{i\in F}\left[\frac{\partial w_j(p^0)}{\partial p_i}\right]\left[\frac{p_i^0}{w_j^0}\right] > 1.$$

The set F can be taken to be the set of indices i (non-empty, by implication (iv)) for which $\beta_{ij} > 0$. This means that an equiproportionate increase in the prices of all commodities with indices

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[11] Each commodity is an enemy of some factor.

[12] This important observation is due to Ethier (1974): every commodity is a friend of some factor.

Given any i, donate by G the set of indices j for which $\beta_{ij} > 0$. This set is non-empty, and a strict subset of $\{1, \ldots, n\}$ by (R). Then, we have, by (11), (18) and (R),

$$1 = \sum_{j=1}^{n}\beta_{ij}\alpha_{ji} < \sum_{j\in G}\beta_{ij}\alpha_{ji} \le \left[\max_{j\in G}\beta_{ij}\right]\sum_{j\in G}\alpha_{ji} < \left[\max_{j\in G}\beta_{ij}\right]$$

[13] So, in general, every factor need not have a commodity friend.

i in the set F, other prices being constant, will increase the *real reward* of factor j.[14]

3.2. A result on sign patterns

Implications (i)–(v), which have been discussed in the literature, follow from (11), (17) and (18). So, a legitimate question is whether these implications exhaust all the restrictions contained in (11), (17) and (18).

In this subsection, we establish a result which shows that there is a restriction on the *sign patterns* of $T \equiv (\beta_{ij})$ that follows from (11), (17) and (18). We describe the result in words before proceeding with a formal discussion. The result shows that the inverse of the distributive share matrix *cannot* be *anti-symmetric* in its sign pattern. This means that one cannot partition the commodities into two groups (I and II) and factors into two groups (A and B), such that all factors in group A benefit (nominally) from all commodity price increases in group I, and simultaneously all factors in group B suffer from all commodity price increases in group II.

To illustrate the nature of the restriction, one may note that (for $n = 4$) the matrix (β_{ij}) *cannot* be of the following form:

$$\begin{bmatrix} * & * & - & - \\ * & * & - & - \\ + & + & * & * \\ + & + & * & * \end{bmatrix}. \tag{19}$$

[Here, and in what follows, a "+" indicates that the entry in the cell is ≥ 0, and a "−"indicates that the entry in the cell is ≤ 0]. In (19), commodities have been partitioned into two groups, I and II, with I consisting of commodities {3, 4} and II consisting of commodities {1, 2}. Factors have been partitioned into two groups, A and B, with A consisting of factors {1, 2} and B consisting of factors {3, 4}. Then the sign pattern in (19) indicates that all factors in group A benefit nominally from all commodity price increases in group I, and all factors in group B suffer from all commodity price increases in group II.

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[14] Every factor has a group of good friends. The pun is, of course, intended. Given any j, denote by F the set of indices i for which $\beta_{ij} > 0$. The set F is non-empty, and a strict subset of $\{1, \ldots, n\}$ by (R). Then, by (11), (17), (18) and (R), we have

$$1 = \sum_{i=1}^{n} \beta_{ij}\alpha_{ji} < \sum_{i \in F} \beta_{ij}\alpha_{ji} \leq \left[\max_{i \in F} \alpha_{ji}\right] \sum_{i \in F} \beta_{ij} < \sum_{i \in F} \beta_{ij}$$

In this proof, I have deliberately refrained from referring to Samuelson's duality theorem, which is usually invoked to establish this result. Instead, I have tried to keep the proof similar to the proof of implication (iii) above.

In particular, in the example described in Section 1, the sign pattern described in (19) occurs. Thus, without consulting any other quantitative details of the matrix in that example, one can conclude that it *cannot* be the inverse of a distributive share matrix (in our maintained framework). That is, the behavior described in the matrix $T = (\beta_{ij})$ in that example cannot be rationalized in terms of our maintained framework, even though it clearly satisfies *all* the restrictions described in (i)–(v) of the last section.[15]

THEOREM 1. *Let* $T = (\beta_{ij})$ *denote the inverse of a distributive share matrix* $S = (\alpha_{ij})$. *Then T cannot satisfy the following condition.*

CONDITION C. *There exist integers k,m, with* $1 \leq k < n$, *and* $1 \leq m < n$, *such that the following inequalities hold simultaneously:*

(i) $\beta_{ij} \leq 0$ *for all* $i = 1, \ldots, k$ *and* $j = m+1, \ldots, n$, *and*
(ii) $\beta_{ij} \geq 0$ *for all* $i = k+1, \ldots, n$ *and* $j = 1, \ldots, m$.

PROOF. Suppose, on the contrary, that $T = (\beta_{ij})$ satisfies Condition C. Then, there exist integers k,m, with $1 \leq k < n$ and $1 \leq m < n$, such that the inverse of the distribution matrix, $T = (\beta_{ij})$ satisfies (i) and (ii) simultaneously.

Focus on the distributive shares of the first industry; that is, on the first column of S. For $i = 1$, we have by (18),

$$\sum_{j=1}^{m} \beta_{ij}\alpha_{j1} + \sum_{j=m+1}^{n} \beta_{ij}\alpha_{j1} = 1. \tag{20}$$

And, for each $i \in \{2, \ldots, k\}$ (if any), we have, by (18)

$$\sum_{j=1}^{m} \beta_{ij}\alpha_{j1} + \sum_{j=m+1}^{n} \beta_{ij}\alpha_{j1} = 0. \tag{21}$$

Using (11) and (i), we know that the second term on the left-hand side of (20) and (21) must be non-positive, and so

$$\begin{bmatrix} \sum_{j=1}^{m} \beta_{1j}\alpha_{j1} \\ \cdots \\ \cdots \\ \sum_{j=1}^{m} \beta_{kj}\alpha_{j1} \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \tag{22}$$

where each vector in (22) has k co-ordinates.

[15] In fact, the matrix $T = (\beta_{ij})$ in the example demonstrates (to use the terminology of Chipman (1969)) strong Stolper–Samuelson properties in response to increase in prices of commodities 1 and 2, and weak Stolper–Samuelson properties in response to increase in prices of commodities 3 and 4.

Define γ to be the column vector in $\mathbb{R}^n$, given by $\gamma = [\alpha_{11}, \ldots, \alpha_{m1}, 0, \ldots, 0]$. That is, it differs from the first column of S in that the last $(n-m)$ entries are zeroes. Then, denoting the n rows of the (β_{ij}) matrix by $(\beta_1, \ldots, \beta_n)$, we can use (22) to write

$$\begin{bmatrix} \beta_1\gamma \\ \cdots \\ \cdots \\ \beta_k\gamma \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m} \beta_{1j}\alpha_{j1} \\ \cdots \\ \cdots \\ \sum_{j=1}^{m} \beta_{kj}\alpha_{j1} \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \tag{23}$$

where each entry in the vector on the left is an inner product of two vectors in $\mathbb{R}^n$.

Using (ii), we also have

$$\begin{bmatrix} \beta_{k+1}\gamma \\ \cdots \\ \beta_n\gamma \end{bmatrix} \geq \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} \tag{24}$$

where each vector in (24) has $(n-k)$ co-ordinates. Thus, combining (23) and (24), we obtain:

$$\begin{bmatrix} \beta_1\gamma \\ \cdots \\ \cdots \\ \beta_n\gamma \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \tag{25}$$

where each vector in (25) has n co-ordinates. Clearly, (25) is the same as writing

$$T\gamma \geq e^1 > 0. \tag{26}$$

Since S is a positive matrix (see (11)), we obtain from (26),

$$S(T\gamma) \geq Se^1 \gg 0. \tag{27}$$

But, since $S(T\gamma) = (ST)\gamma = \gamma$ (by using (18)), (27) implies that $\gamma \gg 0$. This is clearly a contradiction, since the last $(n-m) \geq 1$ entries of γ are zeroes.

Remarks.

(i) The theorem can be derived by using the mathematical result obtained by Johnson *et al.* (1979, Theorem, p. 76) and Fiedler and Grone (1981, Theorem, p. 240). Those results relate to the general problem of characterizing matrices whose inverses are positive matrices, known as the "inverse-positive matrix problem." They are not explicitly concerned with column-stochastic matrices (as we are), but the translation of the results can clearly be made to that sub-class. Johnson *et al.* (1979) use stronger hypotheses, in which (i) and (ii) of Condition C are required to hold with strict inequalities, although their approach indicates that the weaker hypotheses made here suffice. Our proof is slightly different and a bit simpler, and is presented primarily to keep our exposition self-contained.

(ii) Since the numbering of factors and commodities has played no role in our analysis, the (β_{ij}) matrix, *after any independent renumbering of factors and commodities*, cannot exhibit an anti-symmetric sign pattern in the sense described in Condition C of the theorem.

(iii) The theorem would, of course, be valid under any *strengthening* of Condition C. In particular, inequalities (ii) can be strengthened to

$$(\text{ii}')\ \ \beta_{ij} \geq 1 \ \text{ for all } \ i = k+1, \ldots, n \ \text{ and } \ j = 1, \ldots, m,$$

which can be interpreted in terms of the effects of changes in commodity prices on *real* factor rewards.

(iv) From the perspective of the result contained in Theorem 1, the original Stolper–Samuelson result in the 2×2 case can be viewed as one which *rules out* the patterns of behavior of (β_{ij}), where $\beta_{ij} = \left[\partial w_j(p)/\partial p_i\right]\left[p_i/w_j\right]$, given by

$$(\beta_{ij}) = \begin{bmatrix} * & - \\ + & * \end{bmatrix}$$

and its variations, described in remark (ii) above. We elaborate further on the low-dimensional cases (that is, $n = 2$ and $n = 3$) in the next subsection.

3.3. Discussion of low-dimensional cases

The sign-pattern restriction (obtained in Theorem 1) on the inverse of the distributive share matrix is clearly an *additional* restriction that goes beyond the implications (i)–(v) in the case $n = 4$ as our example shows.

We now note that for $n = 2$ and for $n = 3$, it is *not* an additional restriction; that is, for $n = 2$ and $n = 3$, we show that the restriction (R) implies that Condition C cannot hold.

Consider a 2×2 matrix $T = (\beta_{ij})$, which satisfies Condition C. Then, it must be of the form

$$T = (\beta_{ij}) = \begin{bmatrix} * & - \\ + & * \end{bmatrix}. \tag{28}$$

If $\beta_{11} \leq 0$, then row 1 violates (R). However, if $\beta_{11} > 0$, then column 1 violates (R). Thus, in either case, (R) would be violated if Condition C holds.

Consider a 3×3 matrix $T = (\beta_{ij})$, which satisfies Condition C. Then, it can be of several forms. But, it is easy to check that the analysis in all cases can be reduced to the cases depicted in the following two forms:

$$\text{(i)} \;\; T = (\beta_{ij}) = \begin{bmatrix} * & * & - \\ + & + & * \\ + & + & * \end{bmatrix} \;\; \text{or} \;\; \text{(ii)} \;\; T = (\beta_{ij}) = \begin{bmatrix} * & * & - \\ * & * & - \\ + & + & * \end{bmatrix}. \tag{29}$$

Consider (29)(i) first. For (R) to hold, we must clearly have $\beta_{11} < 0$ and $\beta_{12} < 0$. But, then the first row of T would violate (R).

Next, consider (29)(ii). If $\beta_{33} \leq 0$, then the third column of T violates (R). However, if $\beta_{33} > 0$, then the third row of T violates (R).

3.4. A converse result on sign patterns

We now investigate whether the sign-pattern restriction described in Theorem 1, together with (R), exhaust all the restrictions imposed by our framework on the relationship between commodity prices and factor rewards. Qualitatively, it does, as the following result notes.

THEOREM 2. *Let $T = (\beta_{ij})$ be an $n \times n$ non-singular matrix, whose column sums are equal to 1. Suppose T satisfies (R), and does not satisfy Condition C of Theorem 1, even after all possible independent renumberings of its rows and columns. Then, there is an $n \times n$ positive non-singular matrix $S = (\alpha_{ij})$, whose column sums are equal to 1, such that the inverse of S has the same sign pattern as T.*

PROOF. We can apply the theorem of Fiedler and Grone (1981, p. 240)[16] to obtain an $n \times n$ positive non-singular matrix $A = (a_{ij})$; such that the inverse of A (denoted by $B = (b_{ij})$) has the same sign pattern as T.

Define, for each $j \in \{1, \ldots, n\}$:

$$\sigma_j = \sum_{i=1}^{n} a_{ij} \tag{30}$$

and denote by Z the diagonal matrix, with $(1/\sigma_1, \ldots, 1/\sigma_n)$ on its diagonal. Then, AZ is an $n \times n$ positive non-singular matrix, whose column sums are equal to 1, by (30). Define $S = AZ$. Then,

$$S^{-1} = Z^{-1}A^{-1} = Z^{-1}B. \tag{31}$$

[16] Use the equivalence of statements (i) and (ii) in their theorem.

Since Z^{-1} is a diagonal matrix with $(\sigma_1, \ldots, \sigma_n)$ on its diagonal, the sign pattern of S^{-1} is the same as the sign pattern of B, which is the same as the sign pattern of T.

Remark. The matrix S obtained in Theorem 2 can be interpreted as a distributive share matrix, arising from the framework described in Section 2, whose inverse exhibits the sign pattern of T. That is, this distributive share matrix, S, would yield precisely the same qualitative features regarding the effect of commodity prices on factor rewards as indicated by the sign pattern of T. In this sense, the sign pattern of T can be rationalized by the distributive share matrix S.

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